

Differential and falsified sampling expansions *

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Abstract

Differential and falsified sampling expansions $\sum_{k \in \mathbb{Z}^d} c_k \varphi(M^j x + k)$, where M is a matrix dilation, are studied. In the case of differential expansions, $c_k = Lf(M^{-j} \cdot)(-k)$, where L is an appropriate differential operator. For a large class of functions φ , the approximation order of differential expansions was recently studied. Some smoothness of the Fourier transform of φ from this class is required. In the present paper, we obtain similar results for a class of band-limited functions φ with the discontinuous Fourier transform. In the case of falsified expansions, c_k is the mathematical expectation of random integral average of a signal f near the point $M^{-j}k$. To estimate the approximation order of the falsified sampling expansions we compare them with the differential expansions. Error estimations in L_p -norm are given in terms of the Fourier transform of f .

Keywords differential expansion, falsified sampling expansion, approximation order, matrix dilation, Strang-Fix condition.

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1 Introduction

The well-known sampling theorem (Kotel'nikov's or Shannon's formula) states that

$$f(x) = \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \operatorname{sinc}(2^j x + k), \quad \operatorname{sinc}(x) := \frac{\sin \pi x}{\pi x}, \quad (1)$$

for band-limited to $[-2^{j-1}, 2^{j-1}]$ signals (functions) f . This formula is very useful for engineers. It was just Kotel'nikov [18] and Shannon [24] who started to apply this formula for signal processing, respectively in 1933 and 1949. Up to now, an overwhelming diversity of digital signal processing applications and devices are based on it and more than successfully use it. However, mathematicians knew this formula much earlier, actually, it can be found in the papers by Ogura [23] (1920), Whittaker [32] (1915), Borel [2] (1897), and even Cauchy [9] (1841).

Equality (1) holds only for functions $f \in L_2(\mathbb{R})$ whose Fourier transform is supported on $[-2^{j-1}, 2^{j-1}]$. However the right hand side of (1) (the sampling expansion of f) has meaning for every continuous f with a good enough decay. The problem of approximation of f by its sampling

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expansions as $j \rightarrow +\infty$ was studied by many mathematicians. We mention only some of such results. Brown [5] proved that for every $x \in \mathbb{R}$

$$\left| f(x) - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \operatorname{sinc}(2^j x + k) \right| \leq C \int_{|\xi| > 2^{j-1}} |\widehat{f}(\xi)| d\xi, \quad (2)$$

whenever the Fourier transform of f is summable on \mathbb{R} . It is known that the pointwise approximation by sampling expansions does not hold for arbitrary continuous functions f , even compactly supported. Moreover, Trynin [30] proved that there exists a continuous function vanishing outside of $(0, \pi)$ such that its deviation from the sampling expansion diverges at every point $x \in (0, \pi)$. Approximation by sampling expansions in L_p -norm was also actively studied. Bardaro, Butzer, Higgins, Stens, and Vinti [3], [6], proved that

$$\Delta_p := \left\| f - \sum_{k \in \mathbb{Z}} f(-2^{-j}k) \operatorname{sinc}(2^j \cdot + k) \right\|_{L_p(\mathbb{R})} \xrightarrow{j \rightarrow +\infty} 0, \quad 1 \leq p < \infty,$$

for $f \in C \cap \Lambda_p$, where Λ_p consists of f such that

$$\sum_{k \in \mathbb{Z}} |f(x_k)|^p (x_k - x_{k-1}) < \infty$$

for some class of admissible partition $\{x_k\}_{k \in \mathbb{Z}}$ of \mathbb{R} . Also they proved that the Sobolev spaces $W_p^r(\mathbb{R})$, $r \in \mathbb{N}$, are subspaces of Λ_p , and that for every $f \in W_p^r(\mathbb{R})$

$$\Delta_p \leq \frac{C \omega(f^{(r)}, 2^{-j})_p}{2^{-jr}}, \quad (3)$$

where $\omega(\cdot)_p$ is the modulus of continuity in $L_p(\mathbb{R})$, which is a typical estimate for shift invariant spaces. In [4], the same authors studied a generalized sampling approximation, replacing the sinc-function by certain linear combinations of B -splines. For the case $p = \infty$, Butzer, Ries, and Stens [8] proved that if a bounded function φ is such that

$$\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |x - k|^{r+1} |\varphi(x - k)| < \infty$$

for some $r \in \mathbb{N}$, then the estimate

$$\sup_{x \in \mathbb{R}} \left| \sum_{k \in \mathbb{Z}} f\left(\frac{k}{W}\right) \varphi(Wt - k) - f(t) \right| \leq C W^{-r} \omega\left(f^{(r)}, \frac{1}{W}\right) \quad (4)$$

holds for all $f \in C^r(\mathbb{R})$ and $W > 0$ if and only if the Strang-Fix condition of order $r + 1$ is satisfied for φ . The minimal requirement for the convergence (in terms of Sobolev spaces) is $f \in W_p^1(\mathbb{R})$, and, by (3), the minimal approximation order is $o(2^{-j})$. Sickel [25], [26] studied error estimations for Δ_p in terms of Besov-Triebel-Lizorkin spaces. In particular, his results provide the approximation order $2^{-j\gamma}$, $1/p \leq \gamma < 1$, for some functions. The author of [27], [28] investigated approximation by sampling expansions

$$\sum_{k \in \mathbb{Z}} f(-2^{-j}k) \varphi(2^j x + k)$$

for a wide class of band-limited functions φ . For $p \geq 2$, the error analysis was given in terms of the Fourier transform of f . In particular, the approximation order $2^{-j(1/p+\alpha)}$ was found for functions f in Sobolev spaces $W_1^1(\mathbb{R})$ with $f' \in \operatorname{Lip}_{L_1} \alpha$, $\alpha > 0$. In the case $1/p + \alpha > 1$ these estimates give

the same approximation order as (3). Similar results were obtained for the generalized sampling expansions (differential expansions)

$$\sum_{k \in \mathbb{Z}} Lf(2^{-j} \cdot)(-k) \varphi(2^j x + k),$$

where $Lf := \sum_{l=0}^m \alpha_l f^{(l)}$, and a function φ satisfies a special condition of compatibility with L . The approximation order depends on m in this case. An analog of Brown's estimate (2) was proved for such expansions in [28].

Multivariate differential expansions

$$\sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi(M^j x + k), \quad (5)$$

where M is a matrix dilation, were studied in [20]. Error estimations in terms of the Fourier transform of the approximated function were given for a large class of functions φ . In particular, this class contains compactly supported functions, but it does not contain functions with discontinuous Fourier transform.

Differential expansions (5) may be useful for some problems and engineering applications. The analog of Kotelnikov's formula (1) for differential expansions can be used to solve differential equations. The solution can be represented in analytic form which depends only on sampled values of known function for some equations, see Sec. 4 for details. On the other hand, the integral average value of a function f near the point $M^{-j}k$ is close to $Lf(M^{-j} \cdot)(-k)$ for certain operators L . Indeed, if $d = 1$, $M = 2$, then

$$\frac{1}{2^{-j}h} \int_{2^{-j}k}^{2^{-j}(k+h)} f(t) dt \approx \frac{1}{2^{-j}h} \int_0^{2^{-j}h} \sum_{l=0}^N \frac{1}{l!} f^{(l)}(2^{-j}k) t^l dt = \sum_{l=0}^N \frac{1}{(l+1)!} h^l \frac{d^l f(2^{-j} \cdot)}{dx^l}(k).$$

But the latter sum is nothing as $Lf(2^{-j} \cdot)(k)$, where $\alpha_l = \frac{1}{(l+1)!} h^l$. This idea is used in Sec. 5 for error analysis of falsified sampling approximation.

In the present paper we study approximation properties of differential expansions (5) for a class of band-limited functions φ with discontinuous Fourier transform. Error estimations in L_p -norm are given in terms of the Fourier transform of the approximated function f . Analogs of the classical sampling theorem and Brown's inequality (2) are proved. We also study the falsified sampling expansions

$$\sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi(M^j x + k),$$

where M is a matrix dilation,

$$E(f, M^{-j}k) = E(f, M^{-j}k, h, w) = \int_0^\infty du w(u) \frac{1}{\text{meas } M^{-j}B_{h(u)}} \int_{M^{-j}B_{h(u)}} f(M^{-j}k + t) dt,$$

$h(u)$ is a positive function defined on $(0, \infty)$ and u is a random value with probability density w . To estimate the approximation order of falsified sampling expansions we compare them with differential expansions. The error estimations of approximation by differential expansions obtained in [20] as well as new estimations for band-limited functions φ are used.

The paper is organized as follows: in Section 2 we introduce notation and give some basic facts. In Section 3 we study scaling operators $\sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_{jk} \rangle \varphi_{jk}$ and their approximation properties for a class of band-limited functions φ with discontinuous Fourier transform. In Section 4 we study approximation properties of generalized sampling expansions defined by differential operators. In Section 5 we obtain estimates of the approximation order of falsified sampling expansions. In Section 6 we give some examples.

2 Notation and basic facts

\mathbb{N} is the set of positive integers, \mathbb{R} is the set of real numbers, \mathbb{C} is the set of complex numbers. \mathbb{R}^d is the d -dimensional Euclidean space, $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ are its elements (vectors), $(x)_j = x_j$ for $j = 1, \dots, d$, $(x, y) = x_1 y_1 + \dots + x_d y_d$, $|x| = \sqrt{(x, x)}$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$; $B_r = \{x \in \mathbb{R}^d : |x| \leq r\}$, $\mathbb{T}^d = [-\frac{1}{2}, \frac{1}{2}]^d$; \mathbb{Z}^d is the integer lattice in \mathbb{R}^d , $\mathbb{Z}_+^d := \{x \in \mathbb{Z}^d : x \geq \mathbf{0}\}$. If $\alpha, \beta \in \mathbb{Z}_+^d$, $a, b \in \mathbb{R}^d$, we set $[\alpha] = \sum_{j=1}^d \alpha_j$, $\alpha! = \prod_{j=1}^d (\alpha_j!)$,

$$\binom{\beta}{\alpha} = \frac{\alpha!}{\beta!(\alpha - \beta)!}, \quad a^b = \prod_{j=1}^d a_j^{b_j}, \quad D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial x^\alpha} = \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_d} x_d},$$

δ_{ab} is the Kronecker delta.

A real $d \times d$ matrix M whose eigenvalues are bigger than 1 in module is called a dilation matrix. Throughout the paper we consider that such a matrix M is fixed and $m = |\det M|$, M^* denotes the conjugate matrix to M . Since the spectrum of the operator M^{-1} is located in B_r , where $r = r(M^{-1}) := \lim_{j \rightarrow +\infty} \|M^{-j}\|^{1/j}$ is the spectral radius of M^{-1} , and there exists at least one point of the spectrum on the boundary of B_r , we have

$$\|M^{-j}\| \leq C_{M,\vartheta} \vartheta^{-j}, \quad j \geq 0, \quad (6)$$

for every positive number ϑ which is smaller in module than any eigenvalue of M . In particular, we can take $\vartheta > 1$, then

$$\lim_{j \rightarrow +\infty} \|M^{-j}\| = 0. \quad (7)$$

A matrix M is called *isotropic* if it is similar to a diagonal matrix with elements $\lambda_1, \dots, \lambda_d$ that are placed on the main diagonal and $|\lambda_1| = \dots = |\lambda_d|$. Thus, $\lambda_1, \dots, \lambda_d$ are eigenvalues of M and the spectral radius of M is equal to $|\lambda|$, where λ is one of the eigenvalues of M . Note that if the matrix M is isotropic then M^* is isotropic and M^j is isotropic for all $j \in \mathbb{Z}$.

It is well known that for an isotropic matrix M and for any $j \in \mathbb{Z}$ we have

$$C_1 |\lambda|^j \leq \|M^j\| \leq C_2 |\lambda|^j, \quad (8)$$

where λ is one of the eigenvalues of M and the positive constants C_1 and C_2 do not depend on j .

If φ is a function defined on \mathbb{R}^d , we set

$$\varphi_{jk}(x) := m^{j/2} \varphi(M^j x + k), \quad j \in \mathbb{Z}, \quad k \in \mathbb{R}^d.$$

L_p denotes $L_p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. We use W_p^n (or $W_p^n(\mathbb{R}^d)$), $1 \leq p \leq \infty$, $n \in \mathbb{N}$, to denote the Sobolev space on \mathbb{R}^d , i.e. the set of functions whose derivatives up to order n are in L_p , with the usual Sobolev norm.

If f, g are functions defined on \mathbb{R}^d and $f\bar{g} \in L_1$, then $\langle f, g \rangle := \int_{\mathbb{R}^d} f\bar{g}$.

For any function f , we set $f^-(x) := f(-x)$.

If F is a 1-periodic (with respect to each variable) function and $F \in L_1(\mathbb{T}^d)$, then $\widehat{F}(k) = \int_{\mathbb{T}^d} F(x) e^{-2\pi i(k, x)} dx$ is its k -th Fourier coefficient. If $f \in L_1$, then its Fourier transform is $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i(x, \xi)} dx$.

Denote by \mathcal{S} the Schwartz class of functions defined on \mathbb{R}^d . The dual space of \mathcal{S} is \mathcal{S}' , i.e. \mathcal{S}' is the space of tempered distributions. The basic facts from distribution theory can be found, e.g., in [31]. Suppose $f \in \mathcal{S}$, $\varphi \in \mathcal{S}'$, then $\langle \varphi, f \rangle := \langle f, \varphi \rangle := \varphi(f)$. If $\varphi \in \mathcal{S}'$, then $\widehat{\varphi}$ denotes its Fourier transform defined by $\langle \widehat{f}, \widehat{\varphi} \rangle = \langle f, \varphi \rangle$, $f \in \mathcal{S}$. If $\varphi \in \mathcal{S}'$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, then we define φ_{jk} by $\langle f, \varphi_{jk} \rangle = \langle f_{-j, -M^{-j}k}, \varphi \rangle$ for all $f \in \mathcal{S}$.

Given a dilation matrix M and $\delta > 0$, we introduce a special notation for the following integrals if they make sense

$$\mathcal{I}_{j,\gamma,q}^{\text{In}}(g) := \int_{|M^{*-j}\xi| < \delta} |\xi|^{q\gamma} |g(\xi)|^q d\xi, \quad \mathcal{I}_{j,\gamma,q}^{\text{Out}}(g) := \int_{|M^{*-j}\xi| \geq \delta} |\xi|^{q\gamma} |g(\xi)|^q d\xi.$$

A function $\varphi \in L_1$ is said to satisfy the *Strang-Fix condition of order n* if $D^\beta \widehat{\varphi}(k) = 0$, whenever $k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and $[\beta] < n$.

Let $1 \leq p \leq \infty$. Denote by \mathcal{L}_p the set

$$\mathcal{L}_p := \left\{ \varphi \in L_p : \|\varphi\|_{\mathcal{L}_p} := \left\| \sum_{k \in \mathbb{Z}^d} |\varphi(\cdot + k)| \right\|_{L_p(\mathbb{T}^d)} < \infty \right\}.$$

With the norm $\|\cdot\|_{\mathcal{L}_p}$, \mathcal{L}_p is a Banach space. The simple properties are: $\mathcal{L}_1 = L_1$, $\|\varphi\|_p \leq \|\varphi\|_{\mathcal{L}_p}$, $\|\varphi\|_{\mathcal{L}_q} \leq \|\varphi\|_{\mathcal{L}_p}$ for $1 \leq q \leq p \leq \infty$. Therefore, $\mathcal{L}_p \subset L_p$ and $\mathcal{L}_p \subset \mathcal{L}_q$ for $1 \leq q \leq p \leq \infty$. If $\varphi \in L_p$ and compactly supported then $\varphi \in \mathcal{L}_p$ for $p \geq 1$. If φ decays fast enough, i.e. there exist constants $C > 0$ and $\varepsilon > 0$ such that $|\varphi(x)| \leq C(1 + |x|)^{-d-\varepsilon}$ for all $x \in \mathbb{R}^d$, then $\varphi \in \mathcal{L}_\infty$.

The following auxiliary statements will be useful for us.

Proposition 1 ([14]) *Let $1 \leq p \leq \infty$. If $\varphi \in \mathcal{L}_p$ and $a = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_p$, then*

$$\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_p \leq \|\varphi\|_{\mathcal{L}_p} \|a\|_{\ell_p}.$$

Lemma 2 ([20]) *Let $1 \leq q < \infty$, $1/p + 1/q = 1$, $j \in \mathbb{Z}_+$, $\widetilde{\varphi}$ be a tempered distribution whose Fourier transform $\widehat{\widetilde{\varphi}}$ is a function on \mathbb{R}^d such that $|\widehat{\widetilde{\varphi}}(\xi)| \leq C_{\widetilde{\varphi}} |\xi|^N$ for almost all $\xi \notin \mathbb{T}^d$, $N = N(\widetilde{\varphi}) \geq 0$, and $|\widehat{\widetilde{\varphi}}(\xi)| \leq C'_{\widetilde{\varphi}}$ for almost all $\xi \in \mathbb{T}^d$. Suppose $g \in L_q$, $g(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \rightarrow \infty$, where $\varepsilon > 0$; $\gamma \in (N + \frac{d}{p}, N + \frac{d}{p} + \varepsilon)$ for $q \neq 1$, $\gamma = N$ for $q = 1$, and set*

$$G_j(\xi) = G_j(\widetilde{\varphi}, g, \xi) := \sum_{l \in \mathbb{Z}^d} g(M^{*j}(\xi + l)) \overline{\widehat{\widetilde{\varphi}}(\xi + l)}.$$

Then G_j is a 1-periodic function in $L_q(\mathbb{T}^d)$, $\langle g, \widehat{\widetilde{\varphi}}_{jk} \rangle = m^{j/2} \widehat{G}_j(k)$, and for every $\delta \in (0, \frac{1}{2})$

$$\left\| G_j - g(M^{*j} \cdot) \overline{\widehat{\widetilde{\varphi}}} \right\|_{L_q(\mathbb{T}^d)}^q \leq m^{-j} (C_{\gamma, \widetilde{\varphi}})^q \|M^{*-j}\|^{q\gamma} \mathcal{I}_{j,\gamma,q}^{\text{Out}}(g). \quad (9)$$

Lemma 3 ([20]) *Let g and $\widetilde{\varphi}$ be as in Lemma 2. Suppose $2 \leq p \leq \infty$, $\varphi \in \mathcal{L}_p$. Then the series $\sum_{k \in \mathbb{Z}^d} \langle g, \widehat{\widetilde{\varphi}}_{jk} \rangle \varphi_{jk}$ converges unconditionally in L_p .*

3 Scaling Approximation

Scaling operator $\sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\varphi}_{jk} \rangle \varphi_{jk}$ is a good tool of approximation for many appropriate pairs of functions $\varphi, \widetilde{\varphi}$. We are interested in such operators, where $\widetilde{\varphi}$ is a tempered distribution, e.g., the delta-function or a linear combination of its derivatives. In this case the inner product $\langle f, \widetilde{\varphi}_{jk} \rangle$ has meaning only for functions f in \mathcal{S} . To extend the class of functions f one can replace $\langle f, \widetilde{\varphi}_{jk} \rangle$ by $\langle \widehat{f}, \widehat{\widetilde{\varphi}}_{jk} \rangle$. In this case we set

$$Q_j(\varphi, \widetilde{\varphi}, f) = \sum_{k \in \mathbb{Z}^d} \langle \widehat{f}, \widehat{\widetilde{\varphi}}_{jk} \rangle \varphi_{jk}, \quad j \in \mathbb{Z}_+.$$

Approximation properties of such operators for certain classes of distributions $\tilde{\varphi}$ and functions φ were studied in [20]. In particular, the following statement is proved.

Theorem 4 ([20]) *Let $2 \leq p \leq \infty$, $1/p + 1/q = 1$, $N \in \mathbb{Z}_+$, $\gamma \in (N + \frac{d}{p}, N + \frac{d}{p} + \epsilon)$ for $p \neq \infty$, and $\gamma = N$ for $p = \infty$. Suppose*

- (a) *$\tilde{\varphi}$ is a tempered distribution whose Fourier transform $\widehat{\tilde{\varphi}}$ is a function on \mathbb{R}^d such that $|\widehat{\tilde{\varphi}}(\xi)| \leq C_{\tilde{\varphi}}|\xi|^N$ for almost all $\xi \notin \mathbb{T}^d$, $N > 0$, and $|\widehat{\tilde{\varphi}}(\xi)| \leq C'_{\tilde{\varphi}}$ for almost all $\xi \in \mathbb{T}^d$;*
- (b) *$\varphi \in \mathcal{L}_p$ and there exists $B_\varphi > 0$ such that $\sum_{k \in \mathbb{Z}^d} |\widehat{\varphi}(\xi + k)|^q < B_\varphi$ for all $\xi \in \mathbb{R}^d$;*
- (c) *there exist $n \in \mathbb{N}$ and $\delta \in (0, 1/2)$ such that $\widehat{\tilde{\varphi}}\widehat{\varphi}$ is boundedly differentiable up to order n on $\{|\xi| < \delta\}$, $\widehat{\varphi}$ is boundedly differentiable up to order n on $\{|\xi + l| < \delta\}$ for all $l \in \mathbb{Z}^d \setminus \{0\}$; the function $\sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \widehat{\varphi}(\xi + l)|$ is bounded on $\{|\xi| < \delta\}$ for $[\beta] = n$; $D^\beta(1 - \widehat{\tilde{\varphi}}\widehat{\varphi})(0) = 0$ for $[\beta] < n$; the Strang-Fix condition of order n holds for φ ;*
- (d) *$f \in L_p$, $\hat{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-d-\epsilon})$ as $|\xi| \rightarrow \infty$, $\epsilon > 0$.*

Then

$$\|f - Q_j(\varphi, \tilde{\varphi}, f)\|_p^q \leq C_1 \|M^{*-j}\|^{\gamma q} \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f}) + C_2 \|M^{*-j}\|^{nq} \mathcal{I}_{j,n,q}^{\text{In}}(\hat{f}), \quad (10)$$

where C_1 and C_2 do not depend on j and f .

A compability between distribution $\tilde{\varphi}$ and function φ given in item (c) is required for the error of approximation in the latter theorem. This compability is given in terms of the derivatives of $\widehat{\tilde{\varphi}}$, $\widehat{\varphi}$ at the origin up to the order of the Strang-Fix condition of φ . In what follows we will also consider a compability up to an arbitrary order.

Definition 5 *A tempered distribution $\tilde{\varphi}$ and a function φ is said to be strictly compatible if there exists $\delta \in (0, 1/2)$ such that $\widehat{\tilde{\varphi}}(\xi)\widehat{\varphi}(\xi) = 1$ a.e. on $\{|\xi| < \delta\}$ and $\widehat{\varphi}(\xi) = 0$ a.e. on $\{|l - \xi| < \delta\}$ for all $l \in \mathbb{Z} \setminus \{0\}$.*

The class of functions φ considered in Theorem 4 is large, it contains both compactly supported and band-limited functions. However, it does not include functions whose Fourier transform is discontinuous, for example, the function $\prod_{k=1}^d \text{sinc}(x_k)$ is out of consideration. Here we will make up for this omission.

Let $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ denote the class of functions φ given by

$$\varphi(x) = \int_{\mathbb{R}^d} \theta(\xi) e^{2\pi i(x,\xi)} d\xi, \quad (11)$$

where θ is supported on a parallelepiped $S := [a_1, b_1] \times \cdots \times [a_d, b_d]$ and such that $\theta|_S \in C^d(S)$.

Proposition 6 *Let $1 < p < \infty$, $\varphi \in \mathcal{B}$, $f \in L_p$. Then*

$$\left(\sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_{0k} \rangle|^p \right)^{\frac{1}{p}} \leq C_{\varphi,p} \|f\|_p. \quad (12)$$

The proof of Proposition 6 is based on two lemmas. To formulate these lemmas, as well as further results, we need additional notation. Set

$$U_k^0 = \{t \in \mathbb{R} : |t - k| < 1\} \quad \text{and} \quad U_k^1 = \mathbb{R} \setminus U_k^0, \quad k \in \mathbb{Z};$$

if $k \in \mathbb{Z}^d$, $\chi = (\chi_1, \dots, \chi_d) \in \{0, 1\}^d$, then U_k^χ is defined by

$$U_k^\chi = U_{k_1}^{\chi_1} \times \cdots \times U_{k_d}^{\chi_d}.$$

The proof of the lemma below follows easily from the proof of Lemma 4 in [27].

Lemma 7 Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, $u \in \mathbb{R}$. Then

$$\left(\sum_{k \in \mathbb{Z}} \left| \int_{U_k^1} f(t) \frac{e^{2\pi i u(t-k)}}{t-k} dt \right|^p \right)^{\frac{1}{p}} \leq C_p \|f\|_{L_p(\mathbb{R})}.$$

The following lemma gives a proof of Proposition 6 in the case $d = 1$. Since $S = [a_1, b_1]$ if $d = 1$, for convenience, we redenote $[a_1, b_1]$ by $[a, b]$ for this case.

Lemma 8 Let $f \in L_p(\mathbb{R})$, $1 < p < \infty$, and $\varphi \in \mathcal{B}(\mathbb{R})$. Then, for any $\chi \in \{0, 1\}$, we have

$$\left(\sum_{k \in \mathbb{Z}} \left| \int_{U_k^\chi} f(t) \varphi(t-k) dt \right|^p \right)^{\frac{1}{p}} \leq C_{S,p} \|\theta\|_{W_\infty^1(S)} \|f\|_{L_p(\mathbb{R})}. \quad (13)$$

Proof. 1) If $\chi = 0$, then by Hölder's inequality, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \int_{U_k^0} f(t) \varphi(t-k) dt \right|^p &\leq \|\varphi\|_{L_\infty(S)}^p \sum_{k \in \mathbb{Z}} \left(\int_{U_k^0} |f(t)| dt \right)^p \leq \\ &C_{S,p} \|\theta\|_{L_\infty(S)}^p \sum_{k \in \mathbb{Z}} \int_{U_k^0} |f(t)|^p dt \leq C_{S,p} \|\theta\|_{L_\infty(S)}^p \|f\|_{L_p(\mathbb{R})}^p, \end{aligned}$$

which implies (13).

2) Let $\chi = 1$. Using the formula

$$\varphi(x) = \int_a^b \theta(\xi) e^{2\pi i x \xi} d\xi = \frac{\theta(b) e^{2\pi i b x} - \theta(a) e^{2\pi i a x}}{2\pi i x} - \frac{1}{2\pi i x} \int_a^b \theta'(\xi) e^{2\pi i x \xi} d\xi, \quad (14)$$

we obtain

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left| \int_{U_k^1} f(t) \varphi(t-k) dt \right|^p &\leq C_p \sum_{k \in \mathbb{Z}} \left| \int_{U_k^1} f(t) \frac{\theta(b) e^{2\pi i b t} - \theta(a) e^{2\pi i a t}}{t-k} dt \right|^p + \\ &C_p \sum_{k \in \mathbb{Z}} \left| \int_{U_k^1} \frac{f(t)}{t-k} \left(\int_a^b \theta'(\xi) e^{2\pi i (t-k) \xi} d\xi \right) dt \right|^p = I_1 + I_2. \end{aligned} \quad (15)$$

By Lemma 7, we get

$$I_1 \leq C_p \sum_{u \in \{a, b\}} |\theta(u)|^p \sum_{k \in \mathbb{Z}} \left| \int_{U_k^1} f(t) \frac{e^{2\pi i u(t-k)}}{t-k} dt \right|^p \leq C_p \|\theta\|_{L_\infty(S)}^p \|f\|_{L_p(\mathbb{R})}^p. \quad (16)$$

Now, let us consider the sum I_2 . Using Hölder's inequality with $1/p + 1/q = 1$ and Lemma 7, we

derive

$$\begin{aligned}
I_2 &= C_p \sum_{k \in \mathbb{Z}} \left| \int_a^b \theta'(\xi) \int_{U_k^1} f(t) \frac{e^{2\pi i(t-k)\xi}}{t-k} dt d\xi \right|^p \leq \\
&C_p \sum_{k \in \mathbb{Z}} \left(\int_a^b |\theta'(\xi)| \left| \int_{U_k^1} f(t) \frac{e^{2\pi i(t-k)\xi}}{t-k} dt \right| d\xi \right)^p \leq \\
&C_p \sum_{k \in \mathbb{Z}} \left(\int_a^b |\theta'(\xi)|^q d\xi \right)^{\frac{p}{q}} \int_a^b \left| \int_{U_k^1} f(t) \frac{e^{2\pi i(t-k)\xi}}{t-k} dt \right|^p d\xi \leq \\
&C_{S,p} \|\theta'\|_{L_\infty(S)}^p \int_a^b \sum_{k \in \mathbb{Z}} \left| \int_{U_k^1} f(t) \frac{e^{2\pi i(t-k)\xi}}{t-k} dt \right|^p d\xi \leq C_{S,p} \|\theta'\|_{L_\infty(S)}^p \|f\|_{L_p(\mathbb{R})}^p.
\end{aligned} \tag{17}$$

Finally, combining (15)–(17), we get (13) for $\chi = 1$. \diamond

Proof of Proposition 6. For convenience, we introduce the following notation. If $t \in \mathbb{R}^d$, then $\tilde{t} := (t_1, \dots, t_{d-1}) \in \mathbb{R}^{d-1}$. For a function g of d variables t_1, \dots, t_{d-1} and s , we set $g_s(\tilde{t}) = g(t_1, \dots, t_{d-1}, s)$. Let also $\psi_{\tilde{x}}(\eta) = \mathcal{F}^{-1} \theta_\eta(\tilde{x})$ and $\tilde{S} = [a_1, b_1] \times \dots \times [a_{d-1}, b_{d-1}]$.

We have

$$\sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_{0k} \rangle|^p = \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^d} f(t) \varphi(t-k) dt \right|^p \leq C_p \sum_{\chi \in \{0,1\}^d} I^\chi, \tag{18}$$

where

$$I^\chi = \sum_{k \in \mathbb{Z}} \left| \int_{U_k^\chi} f(t) \varphi(t-k) dt \right|^p.$$

Thus, to prove (12) it is enough to show that

$$I^\chi \leq C_{S,p} \|\theta\|_{W_\infty^d(S)}^p \|f\|_p^p \quad \text{for any } \chi \in \{0,1\}^d. \tag{19}$$

We prove (19) by induction on d . For $d = 1$, the inequality (19) was proved in Lemma 8. To prove the inductive step $d-1 \rightarrow d$, we assume that for any $g \in L_p(\mathbb{R}^{d-1})$ and $\tilde{\varphi} \in \mathcal{B}(\mathbb{R}^{d-1})$ (more precisely $\tilde{\varphi} = \mathcal{F}^{-1} \tilde{\theta}$, where $\tilde{\theta}$ is the same as in (11) with \tilde{S} in place of S), we have

$$\sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \left| \int_{U_{\tilde{k}}^{\tilde{\chi}}} g(\tilde{t}) \tilde{\varphi}(\tilde{t} - \tilde{k}) d\tilde{t} \right|^p \leq C_{\tilde{S},p} \|\tilde{\theta}\|_{W_\infty^{d-1}(\tilde{S})}^p \|g\|_{L_p(\mathbb{R}^{d-1})}^p. \tag{20}$$

For any $\chi \in \{0,1\}^d$, we can write $\chi = (\tilde{\chi}, \chi_d)$ and

$$I^\chi = \sum_{\chi_d \in \{0,1\}} I^{(\tilde{\chi}, \chi_d)}.$$

Let us estimate $I^{(\tilde{\chi}, \chi_d)}$ for $\chi_d = 0$ and $\chi_d = 1$.

1) For $\chi_d = 0$, we obtain

$$I^{(\tilde{\chi}, 0)} = \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_l^0} \int_{U_{\tilde{k}}^{\tilde{\chi}}} f_s(\tilde{t}) \varphi(\tilde{t} - \tilde{k}, s-l) d\tilde{t} ds \right|^p. \tag{21}$$

Using the above notation and formula (14), we get

$$\begin{aligned}\varphi(x) &= \mathcal{F}^{-1}\theta(x) = \int_{a_d}^{b_d} \psi_{\tilde{x}}(\eta) e^{2\pi i x_d \eta} d\eta = \\ &= \frac{\psi_{\tilde{x}}(b_d) e^{2\pi i b_d x_d} - \psi_{\tilde{x}}(a_d) e^{2\pi i a_d x_d}}{2\pi i x_d} - \frac{1}{2\pi i x_d} \mathcal{F}^{-1}\psi'_{\tilde{x}}(x_d).\end{aligned}\tag{22}$$

By (22) and Hölder's inequality, we derive

$$\begin{aligned}\left| \int_{U_l^0} \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \int_{a_d}^{b_d} \psi_{\tilde{t}-\tilde{k}}(\eta) e^{2\pi i(s-l)\eta} d\eta d\tilde{t} ds \right|^p &\leq \left(\int_{a_d}^{b_d} \int_{U_l^0} \left| \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \psi_{\tilde{t}-\tilde{k}}(\eta) d\tilde{t} \right| ds d\eta \right)^p \\ &= \left(\int_{a_d}^{b_d} \int_{U_l^0} ds d\eta \right)^{p-1} \int_{a_d}^{b_d} \int_{U_l^0} \left| \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \psi_{\tilde{t}-\tilde{k}}(\eta) d\tilde{t} \right|^p ds d\eta = (2(b_d - a_d))^{p-1} \int_{a_d}^{b_d} \int_{U_l^0} |F_{\tilde{k},\eta}(s)|^p ds d\eta,\end{aligned}\tag{23}$$

where

$$F_{\tilde{k},\eta}(s) = \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \psi_{\tilde{t}-\tilde{k}}(\eta) d\tilde{t}.\tag{24}$$

Next, combining (21) and (23) and using the induction hypothesis (20), we get

$$\begin{aligned}I^{(\tilde{x},0)} &\leq C_{S,p} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \int_{a_d}^{b_d} \int_{U_l^0} |F_{\tilde{k},\eta}(s)|^p ds d\eta \leq C_{S,p} \int_{a_d}^{b_d} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \|F_{\tilde{k},\eta}\|_{L_p(\mathbb{R})}^p d\eta \leq \\ &= C_{S,p} \int_{a_d}^{b_d} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \left| \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \mathcal{F}^{-1}\theta_{\eta}(\tilde{t}-\tilde{k}) d\tilde{t} \right|^p ds d\eta \leq \\ &= C_{S,p} \int_{a_d}^{b_d} \|\theta_{\eta}\|_{W_{\infty}^{d-1}(\tilde{S})}^p d\eta \int_{\mathbb{R}} \|f_s\|_{L_p(\mathbb{R}^{d-1})}^p ds \leq C_{S,p} \|\theta\|_{W_{\infty}^d(S)}^p \|f\|_p^p.\end{aligned}\tag{25}$$

2) Now we consider the case $\chi_d = 1$. We have

$$I^{(\tilde{x},1)} = \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_l^1} \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \varphi(\tilde{t}-\tilde{k}, s-l) d\tilde{t} ds \right|^p.\tag{26}$$

By (22), we get

$$I^{(\tilde{x},1)} \leq C_p(I_1 + I_2),\tag{27}$$

where

$$I_1 = \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_l^1} \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \frac{\psi_{\tilde{t}-\tilde{k}}(b_d) e^{2\pi i b_d(s-l)} - \psi_{\tilde{t}-\tilde{k}}(a_d) e^{2\pi i a_d(s-l)}}{s-l} d\tilde{t} ds \right|^p$$

and

$$I_2 = \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_l^1} \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \frac{\mathcal{F}^{-1}\psi'_{\tilde{t}-\tilde{k}}(s-l)}{s-l} d\tilde{t} ds \right|^p.\tag{28}$$

Using the function (24) and Lemma 7, we obtain

$$I_1 \leq C_p \sum_{u \in \{a_d, b_d\}} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_l^1} \frac{F_{\tilde{k},u}(s) e^{2\pi i u(s-l)}}{s-l} ds \right|^p \leq C_p \sum_{u \in \{a_d, b_d\}} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \|F_{\tilde{k},u}\|_{L_p(\mathbb{R})}^p. \quad (29)$$

Now, using the induction hypothesis (20), for any $u \in \mathbb{R}$, we derive

$$\begin{aligned} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \|F_{\tilde{k},u}\|_{L_p(\mathbb{R})}^p &= \int_{\mathbb{R}} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \left| \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \mathcal{F}^{-1} \theta_u(\tilde{t} - \tilde{k}) d\tilde{t} \right|^p ds \leq \\ &C_{\tilde{S},p} \|\theta_u\|_{W_{\infty}^{d-1}(\tilde{S})}^p \int_{\mathbb{R}} \|f_s\|_{L_p(\mathbb{R}^{d-1})}^p ds. \end{aligned} \quad (30)$$

Hence, combining (29) and (30), we get

$$I_1 \leq C_{\tilde{S},p} \sum_{u \in \{a_d, b_d\}} \|\theta_u\|_{W_{\infty}^{d-1}(\tilde{S})}^p \|f\|_p^p \leq C_{S,p} \|\theta\|_{W_{\infty}^d(S)}^p \|f\|_p^p. \quad (31)$$

Let us consider I_2 . Denoting

$$F_{\tilde{k},\eta}^*(s) = \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \psi'_{\tilde{t}-\tilde{k}}(\eta) d\tilde{t}$$

and using Hölder's inequality, we obtain

$$\begin{aligned} \left| \int_{U_l^1} \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \frac{\mathcal{F}^{-1} \psi'_{\tilde{t}-\tilde{k}}(s-l)}{s-l} d\tilde{t} ds \right|^p &\leq \left(\int_{a_d}^{b_d} \left| \int_{U_l^1} \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \frac{\psi'_{\tilde{t}-\tilde{k}}(\eta) e^{2\pi i(s-l)\eta}}{s-l} d\tilde{t} ds \right| d\eta \right)^p \leq \\ &(b_d - a_d)^{p-1} \int_{a_d}^{b_d} \left| \int_{U_l^1} F_{\tilde{k},\eta}^*(s) \frac{e^{2\pi i(s-l)\eta}}{s-l} ds \right|^p d\eta. \end{aligned} \quad (32)$$

Thus, combining (28) and (32), using Lemma 7, and the induction hypothesis (20), we derive

$$\begin{aligned} I_2 &\leq C_{S,p} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \int_{a_d}^{b_d} \left| \int_{U_l^1} F_{\tilde{k},\eta}^*(s) \frac{e^{2\pi i(s-l)\eta}}{s-l} ds \right|^p d\eta = \\ &C_{S,p} \int_{a_d}^{b_d} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \sum_{l \in \mathbb{Z}} \left| \int_{U_l^1} F_{\tilde{k},\eta}^*(s) \frac{e^{2\pi i(s-l)\eta}}{s-l} ds \right|^p d\eta \leq \\ &C_{S,p} \int_{a_d}^{b_d} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \|F_{\tilde{k},\eta}^*\|_{L_p(\mathbb{R})}^p d\eta = \\ &C_{S,p} \int_{a_d}^{b_d} \int_{\mathbb{R}} \sum_{\tilde{k} \in \mathbb{Z}^{d-1}} \left| \int_{U_{\tilde{k}}^{\tilde{x}}} f_s(\tilde{t}) \mathcal{F}^{-1} \frac{\partial}{\partial \eta} \theta_{\eta}(\tilde{t} - \tilde{k}) d\tilde{t} \right|^p ds d\eta \leq \\ &C_{S,p} \int_{a_d}^{b_d} \left\| \frac{\partial}{\partial \eta} \theta_{\eta} \right\|_{W_{\infty}^{d-1}(\tilde{S})}^p d\eta \int_{\mathbb{R}} \|f_s\|_{L_p(\mathbb{R}^{d-1})}^p ds \leq C_{S,p} \|\theta\|_{W_{\infty}^d(S)}^p \|f\|_p^p. \end{aligned} \quad (33)$$

Finally, combining (27), (31), and (33), we get (19) for any $\chi = (\chi_1, \dots, \chi_{d-1}, 1)$. This and (25) prove the proposition. \diamond

Proposition 9 *Let $1 < p < \infty$, $\varphi \in \mathcal{B}$, $a = \{a_k\}_{k \in \mathbb{Z}^d} \in \ell_p$. Then*

$$\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_p \leq C_{\varphi, p} \|a\|_{\ell_p}.$$

Proof. Since

$$\left\| \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k} \right\|_p = \left| \left\langle \sum_{k \in \mathbb{Z}^d} a_k \varphi_{0k}, f \right\rangle \right| = \left| \sum_{k \in \mathbb{Z}^d} a_k \langle \varphi_{0k}, f \rangle \right|,$$

where $f \in L_q$, $\|f\|_q \leq 1$, $1/p + 1/q = 1$, the statement follows immediately from Proposition 6 and Hölder's inequality. \diamond

Corollary 10 *Let $2 \leq p < \infty$, $1/p + 1/q = 1$, $g \in L_q$, and $\widetilde{\varphi}$ be as in Lemma 2, $\varphi \in \mathcal{B}$. Then the series $\sum_{k \in \mathbb{Z}^d} \langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle \varphi_{jk}$ converges unconditionally in L_p and*

$$\left\| \sum_{k \in \mathbb{Z}^d} \langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle \varphi_{jk} \right\|_p \leq C_{\varphi, q} m^{\frac{j}{2} - \frac{j}{p}} \left(\sum_{k \in \mathbb{Z}^d} |\langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle|^p \right)^{\frac{1}{p}}. \quad (34)$$

Proof. Because of Lemma 2 and the Hausdorff-Young inequality, we have

$$\left(\sum_{k \in \mathbb{Z}^d} |\langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle|^p \right)^{\frac{1}{p}} = m^{\frac{j}{2}} \left(\sum_{k \in \mathbb{Z}^d} |\widehat{G}_j(k)|^p \right)^{\frac{1}{p}} \leq m^{\frac{j}{2}} \|G_j\|_{L_q(\mathbb{T}^d)} < \infty, \quad (35)$$

where G_j is a function from Lemma 2. By Proposition 9, we can state that for every finite subset Ω of \mathbb{Z}^d

$$\left\| \sum_{k \in \Omega} \langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle \varphi_{jk} \right\|_p = m^{\frac{j}{2} - \frac{j}{p}} \left\| \sum_{k \in \Omega} \langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle \varphi_{0k} \right\|_p \leq m^{\frac{j}{2} - \frac{j}{p}} C_{\varphi, q} \left(\sum_{k \in \Omega} |\langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle|^p \right)^{\frac{1}{p}}.$$

The series $\sum_{k \in \mathbb{Z}^d} |\langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle|^p$ is convergent, which yields that $\sum_{k \in \mathbb{Z}^d} \langle g, \widehat{\widetilde{\varphi}_{jk}} \rangle \varphi_{jk}$ converges unconditionally. Similarly we obtain (34) \diamond

Now we are ready to study approximation properties of the operators

$$Q_j(\varphi, \widetilde{\varphi}, f) = \sum_{k \in \mathbb{Z}^d} \langle \widehat{f}, \widehat{\widetilde{\varphi}_{jk}} \rangle \varphi_{jk},$$

where $\varphi \in \mathcal{B}$. The following theorem is a counterpart of Theorem 4 for such functions φ .

Theorem 11 *Let $2 \leq p < \infty$, $1/p + 1/q = 1$, $N \in \mathbb{Z}_+$, $\gamma \in (N + \frac{d}{p}, N + \frac{d}{p} + \epsilon)$. Suppose*

- (a) $\widetilde{\varphi}$ is a tempered distribution whose Fourier transform $\widehat{\widetilde{\varphi}}$ is a function on \mathbb{R}^d such that $|\widehat{\widetilde{\varphi}}(\xi)| \leq C_{\widetilde{\varphi}} |\xi|^N$ for almost all $\xi \notin \mathbb{T}^d$ and $|\widehat{\widetilde{\varphi}}(\xi)| \leq C'_{\widetilde{\varphi}}$ for almost all $\xi \in \mathbb{T}^d$;

- (b) $\varphi \in \mathcal{B}$;
- (c) $\tilde{\varphi}$ and φ are strictly compatible;
- (d) $f \in L_p$, $\hat{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$.

Then

$$\|f - Q_j(\varphi, \tilde{\varphi}, f)\|_p^q \leq C \|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\|, \quad (36)$$

where C does not depend on j and f .

Proof. Throughout the proof we denote by C_1, C_2, \dots different constants which do not depend on j and f . It follows from (34), (35), and Lemma 2 with $g = \hat{f}$ that

$$\begin{aligned} \|Q_j(\varphi, \tilde{\varphi}, f)\|_p &\leq m^{\frac{j}{q}} C_{\varphi,q} \|G_j\|_{L_q(\mathbb{T}^d)} \leq C_{\varphi,q} m^{\frac{j}{q}} \left(m^{-j} (C_{\gamma,\tilde{\varphi}})^q \|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\| \right)^{1/q} + \\ &\quad C_{\varphi,q} m^{\frac{j}{q}} \left(\int_{\mathbb{T}^d} |\hat{f}(M^{*j}\xi) \tilde{\varphi}(\xi)|^q d\xi \right)^{1/q} \leq C_1 \left(\left(\|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\| \right)^{1/q} + \|\hat{f}\|_q \right). \end{aligned}$$

Since f^- coincides with \hat{f} a.e. due to the du Bois-Reymond lemma, applying the Hausdorff-Young inequality, we have $\|f\|_p = \|f^-\|_p \leq \|\hat{f}\|_q$, which yields

$$\begin{aligned} \|f - Q_j(\varphi, \tilde{\varphi}, f)\|_p &\leq C_2 \left(\left(\|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\| \right)^{1/q} + \|\hat{f}\|_q \right) \leq \\ &\quad C_2 \left(\|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\| + \int_{|M^{*-j}\xi| < \delta} |\hat{f}(\xi)|^q d\xi + \delta^{-q\gamma} \int_{|M^{*-j}\xi| \geq \delta} |M^{*-j}\xi|^{q\gamma} |\hat{f}(\xi)|^q d\xi \right)^{1/q} \leq \\ &\quad C_3 \left(\|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\| + \int_{|M^{*-j}\xi| < \delta} |\hat{f}(\xi)|^q d\xi \right)^{1/q}, \quad (37) \end{aligned}$$

where δ is from Definition 5.

For any compact set $K \subset \mathbb{R}^d$, the function \hat{f} can be approximated in $L_q(K)$ by infinitely smooth functions supported on K . So, given j , one can find a function F_j such that $\text{supp } \hat{F}_j \subset \{|M^{*-j}\xi| < \delta\}$, $\hat{F}_j \in C^\infty(\mathbb{R}^d)$, and

$$\int_{|M^{*-j}\xi| < \delta} |\hat{f}(\xi) - \hat{F}_j(\xi)|^q d\xi \leq \|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\|.$$

Combining this with (37), where f is replaced by $f - F_j$, and taking into account that $\mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f}) = \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f} - \hat{F}_j)$, we have

$$\|(f - F_j) - Q_j(\varphi, \tilde{\varphi}, f - F_j)\|_p^q \leq \|M^{*-j}\| \|\gamma^q \mathcal{I}_{j,\gamma,q}^{\text{Out}}(\hat{f})\|. \quad (38)$$

Due to Carleson's theorem and Lemma 2, we have

$$\sum_{k \in \mathbb{Z}^d} \langle \hat{F}_j, \widehat{\tilde{\varphi}_{jk}} \widehat{\varphi_{jk}} \rangle(\xi) = \sum_{l \in \mathbb{Z}^d} \hat{F}_j(\xi + M^{*j}l) \overline{\widehat{\tilde{\varphi}}(M^{*-j}\xi + l)} \widehat{\varphi}(M^{*-j}\xi).$$

If $l \neq 0$ and $F_j(\xi + M^{*j}l) \neq 0$, then $|M^{*-j}\xi + l| < \delta$ and hence $\widehat{\varphi}(M^{*-j}\xi) = 0$. So,

$$\sum_{k \in \mathbb{Z}^d} \langle \hat{F}_j, \widehat{\tilde{\varphi}_{jk}} \widehat{\varphi_{jk}} \rangle(\xi) = \hat{F}_j(\xi),$$

which yields that $Q_j(\varphi, \tilde{\varphi}, F_j) = F_j$. It follows that

$$\|f - Q_j(\varphi, \tilde{\varphi}, f)\|_p = \|f - F_j - Q_j(\varphi, \tilde{\varphi}, f - F_j)\|_p.$$

Together with (38) this yields (36). \diamond

Corollary 12 *If, under assumptions (a) – (c) of Theorem 11, a function f is such that its Fourier transform is supported in $\{\xi : |M^{*-j}\xi| < \delta\}$, where δ is from Definition 5, then*

$$f = Q_j(\varphi, \tilde{\varphi}, f) \quad \text{a.e.}$$

The latter equality is a generalization of (1).

4 Differential expansions

Consider a differential operator L defined by

$$Lf := \sum_{[\beta] \leq N} a_\beta D^\beta f, \quad a_\beta \in \mathbb{C}, \quad a_0 \neq 0, \quad (39)$$

where $N \in \mathbb{Z}_+$. The action of the operator D^β is associated with the action of the corresponding derivative of the δ -function. In more detail, let f be such that $\int_{\mathbb{R}^d} (1 + |\xi|)^{N+\alpha} |\hat{f}(\xi)| d\xi < \infty$, $\alpha > 0$, which implies that f is continuously differentiable on \mathbb{R}^d up to the order N . Then for $[\beta] \leq N$

$$\begin{aligned} D^\beta f(M^{-j}\cdot)(-k) &= (-1)^{[\beta]} m^j \int_{\mathbb{R}^d} \hat{f}(M^{*j}\xi) (-2\pi i \xi)^\beta e^{-2\pi i(k, \xi)} d\xi = \\ &= (-1)^{[\beta]} \int_{\mathbb{R}^d} \hat{f}(\xi) \overline{(2\pi i M^{*-j}\xi)^\beta e^{2\pi i(M^{-j}k, \xi)}} d\xi = (-1)^{[\beta]} m^{j/2} \langle \hat{f}, \widehat{D^\beta \delta_{jk}} \rangle. \end{aligned}$$

If now $\tilde{\varphi} = \sum_{[\beta] \leq N} \overline{a_\beta} (-1)^{[\beta]} D^\beta \delta$ (we say that $\tilde{\varphi}$ is associated with L), then

$$m^{-j/2} Lf(M^{-j}\cdot)(-k) = \langle \hat{f}, \widehat{\tilde{\varphi}_{jk}} \rangle, \quad k \in \mathbb{Z}^d.$$

Hence,

$$Q_j(\varphi, \tilde{\varphi}, f) = m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j}\cdot)(-k) \varphi_{jk}. \quad (40)$$

We are interested in the error of approximation of a function f by these operators.

Theorem 13 ([20]) *Let $2 \leq p \leq \infty$, $1/p + 1/q = 1$, a differential operator L be defined by (39). Suppose*

- (a) $\tilde{\varphi}$ is the distribution associated with L ;
- (b) $\varphi \in \mathcal{L}_p$ and there exists $B_\varphi > 0$ such that $\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q < B_\varphi$ for all $\xi \in \mathbb{R}^d$;
- (c) there exist $n \in \mathbb{N}$ and $\delta \in (0, 1/2)$ such that $\widehat{\tilde{\varphi}}$ is boundedly differentiable up to order n on $\{|\xi| < \delta\}$, $\widehat{\varphi}$ is boundedly differentiable up to order n on $\{|\xi + l| < \delta\}$ for all $l \in \mathbb{Z}^d \setminus \{0\}$; the function $\sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \widehat{\varphi}(\xi + l)|$ is bounded on $\{|\xi| < \delta\}$ for $[\beta] = n$; $D^\beta (1 - \widehat{\tilde{\varphi}})(0) = 0$ for $[\beta] < n$; the Strang-Fix condition of order n holds for φ ;
- (d) $f \in L_p$, $\hat{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$.

Then the following statements hold:

(A) if $n < N + \frac{d}{p} + \varepsilon$, then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi_{jk} \right\|_p \leq C \vartheta^{-jn}$$

for every positive number ϑ which is smaller in module than any eigenvalue of M and some C which does not depend on j ;

(B) if M is an isotropic matrix and λ is its eigenvalue then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi_{jk} \right\|_p \leq \begin{cases} C|\lambda|^{-j(N+\frac{d}{p}+\varepsilon)} & \text{if } n > N + \frac{d}{p} + \varepsilon \\ C(j+1)^{1/q} |\lambda|^{-jn} & \text{if } n = N + \frac{d}{p} + \varepsilon, \\ C|\lambda|^{-jn} & \text{if } n < N + \frac{d}{p} + \varepsilon \end{cases}, \quad (41)$$

where C does not depend on j ;

(C) if $\tilde{\varphi}$ and φ are strictly compatible, then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi_{jk} \right\|_p \leq C \vartheta^{-j(N+\frac{d}{p}+\varepsilon)}$$

for every positive number ϑ which is smaller in module than any eigenvalue of M and some C which does not depend on j .

Now we prove similar results for the class of functions $\varphi \in \mathcal{B}$.

Theorem 14 Let $2 \leq p < \infty$, $1/p + 1/q = 1$, a differential operator L be defined by (39). Suppose

- (a) $\tilde{\varphi}$ is the distribution associated with L ;
- (b) $\varphi \in \mathcal{B}$;
- (c) $\tilde{\varphi}$ and φ are strictly compatible;
- (d) $f \in L_p$, $\hat{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$.

Then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi_{jk} \right\|_p \leq C \vartheta^{-j(N+\frac{d}{p}+\varepsilon)}, \quad (42)$$

where C does not depend on j .

Proof. Since, by (40), all conditions of Theorem 11 are satisfied for $\tilde{\varphi}$, φ , (42) follows from (36) and the next lemma. \diamond

Lemma 15 Let $1 \leq q < \infty$, $1/p + 1/q = 1$, $N \in \mathbb{Z}_+$, $\varepsilon > 0$, $g \in L_q$, $g(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\gamma < N + \frac{d}{p} + \varepsilon$. Then

$$\|M^{*-j}\|^{\gamma q} \mathcal{I}_{j,\gamma,q}^{\text{Out}}(g) \leq C \vartheta^{-jq(N+\frac{d}{p}+\varepsilon)},$$

where ϑ is any positive number which is smaller in module than any eigenvalue of M , C does not depend on j .

Proof. Throughout the proof we denote by C_1, C_2, \dots different constants which do not depend on j . Since there exists $A > 0$ such that $|g(\xi)| \leq C_1 |\xi|^{-N-d-\varepsilon}$ for any $|\xi| > A$ and the set $\{|M^{*-j}\xi| \geq \delta\}$ is a subset of $\{|\xi| \geq \delta/\|M^{*-j}\|\}$, we have

$$\|M^{*-j}\|^{\gamma q} \mathcal{I}_{j,\gamma,q}^{\text{Out}}(g) = \|M^{*-j}\|^{\gamma q} \int_{|M^{*-j}\xi| \geq \delta} |\xi|^{\gamma q} |g(\xi)|^q d\xi \leq C_1 \|M^{*-j}\|^{\gamma q} \int_{|\xi| \geq \delta/\|M^{*-j}\|} \frac{d\xi}{|\xi|^{(N+d+\varepsilon-\gamma)q}}$$

for all $j > j_0$, where $j_0 \in \mathbb{Z}$ is such that $\frac{\delta}{\|M^{*-j_0}\|} > A$. Using general polar coordinates with $\rho := |\xi|$ and taking into account that $(N+d+\varepsilon-\gamma)q > d$, we obtain

$$\int_{|\xi| \geq \delta/\|M^{*-j}\|} \frac{d\xi}{|\xi|^{(N+d+\varepsilon-\gamma)q}} \leq C_2 \int_{\delta/\|M^{*-j}\|}^{+\infty} \frac{1}{\rho^{(N+d+\varepsilon-\gamma)q-d+1}} d\rho \leq C_3 \|M^{*-j}\|^{q(N+\frac{d}{p}+\varepsilon-\gamma)},$$

and, by (6),

$$\|M^{*-j}\|^{\gamma q} \mathcal{I}_{j,\gamma,q}^{\text{Out}}(g) \leq C_4 \|M^{*-j}\|^{q(N+\frac{d}{p}+\varepsilon)} \leq C \vartheta^{-jq(N+\frac{d}{p}+\varepsilon)}. \quad \diamond$$

Theorem 14 says nothing about $p = \infty$. Now we consider this case and prove a generalization of Brown's inequality (2).

Theorem 16 *Let a differential operator L be defined by (39). Suppose*

- (a) $\tilde{\varphi}$ is the distribution associated with L ;
- (b) $\varphi \in \mathcal{B}$;
- (c) $\tilde{\varphi}$ and φ are strictly compatible;
- (d) $f \in C(\mathbb{R}^d)$, $\hat{f} \in L$, $\hat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$.

Then for every $x \in \mathbb{R}^d$ and $j \in \mathbb{Z}$ the series

$$\sum_{k \in \mathbb{Z}^d} Lf(M^{-j}\cdot)(-k) \varphi_{jk}(x),$$

considered as the limit of cubic partial sums, converges, and

$$\left| f(x) - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j}\cdot)(-k) \varphi_{jk}(x) \right| \leq C \|M^{*-j}\|^N \int_{|M^{*-j}\xi| \geq \delta} |\xi|^N |\hat{f}(\xi)| d\xi \leq C' \vartheta^{-j(N+\varepsilon)}, \quad (43)$$

where C does not depend on f , j , and x ; C' does not depend on j and x ; ϑ is a positive number which is smaller in module than any eigenvalue of M .

Proof. Let φ be given by (11), $x \in \mathbb{R}^d$. Set $\Theta(\xi) := \sum_{s \in \mathbb{Z}^d} \theta(\xi + s) e^{2\pi i(x, \xi + s)}$. By the Poisson summation formula, Θ is a summable 1-periodic (with respect to each variable) function and its n -th Fourier coefficient is

$$\hat{\Theta}(n) = \int_{\mathbb{R}^d} \theta(\xi) e^{-2\pi i(n-x, \xi)} d\xi = \hat{\theta}(n-x) = \varphi(x-n).$$

Since Θ is a bounded function, the cubic partial Fourier sums are uniformly bounded in L_∞ -norm, and the corresponding Fourier series converges to the function almost everywhere. Using this and

Lebesgue's dominated convergence theorem, for every $\beta \in \mathbb{Z}^d$, $[\beta] \leq N$, we derive

$$\begin{aligned} \lim_{M \rightarrow \infty} \sum_{\|n\|_\infty \leq M} D^\beta f(-n) \varphi(x+n) &= \lim_{M \rightarrow \infty} \int_{\mathbb{R}^d} \sum_{\|n\|_\infty \leq M} \varphi(x+n) e^{-2\pi i(n,\xi)} (2\pi i \xi)^\beta \widehat{f}(\xi) d\xi = \\ \int_{\mathbb{R}^d} \lim_{M \rightarrow \infty} \sum_{\|n\|_\infty \leq M} \varphi(x-n) e^{2\pi i(n,\xi)} (2\pi i \xi)^\beta \widehat{f}(\xi) d\xi &= \int_{\mathbb{R}^d} e^{2\pi i(x,\xi)} \sum_{s \in \mathbb{Z}^d} \theta(\xi+s) e^{2\pi i(x,s)} (2\pi i \xi)^\beta \widehat{f}(\xi) d\xi. \end{aligned}$$

Replacing x by $M^j x$ and f by $f(M^{-j} \cdot)$, after a change of variable, we obtain

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d} D^\beta f(M^{-j} \cdot)(-n) \varphi_{jn}(x) &= \lim_{M \rightarrow \infty} \sum_{\|n\|_\infty \leq M} D^\beta f(M^{-j} \cdot)(-n) \varphi_{jn}(x) = \\ m^{j/2} \int_{\mathbb{R}^d} e^{2\pi i(x,\xi)} \sum_{s \in \mathbb{Z}^d} \theta(M^{*-j} \xi + s) e^{2\pi i(M^j x, s)} (2\pi i M^{*-j} \xi)^\beta \widehat{f}(\xi) d\xi. \end{aligned}$$

This yields

$$\begin{aligned} m^{-j/2} \sum_{n \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-n) \varphi_{jn}(x) &= \\ \int_{\mathbb{R}^d} e^{2\pi i(x,\xi)} \sum_{s \in \mathbb{Z}^d} \theta(M^{*-j} \xi + s) e^{2\pi i(M^j x, s)} \sum_{[\beta] \leq N} \alpha_\beta (2\pi i M^{*-j} \xi)^\beta \widehat{f}(\xi) d\xi. \end{aligned} \quad (44)$$

Set

$$f_1(x) = \int_{|M^{*-j} \xi| \leq \delta} \widehat{f}(\xi) e^{2\pi i(x,\xi)} d\xi, \quad f_2(x) = f(x) - f_1(x).$$

If $|M^{*-j} \xi| \leq \delta$, then

$$\sum_{s \in \mathbb{Z}^d} \theta(M^{*-j} \xi + s) e^{2\pi i(M^j x, s)} = \widehat{\varphi}(M^{*-j} \xi)$$

Taking into account that $\widehat{\varphi}(M^{*-j} \xi) \overline{\widehat{\varphi}(M^{*-j} \xi)} = 1$, we have

$$\sum_{s \in \mathbb{Z}^d} \theta(M^{*-j} \xi + s) e^{2\pi i(M^j x, s)} \sum_{[\beta] \leq N} \alpha_\beta (2\pi i M^{*-j} \xi)^\beta = 1.$$

Hence, it follows from (44) that

$$m^{-j/2} \sum_{n \in \mathbb{Z}^d} Lf_1(M^{-j} \cdot)(-n) \varphi_{jn}(x) = \int_{|M^{*-j} \xi| \leq \delta} e^{2\pi i x \xi} \widehat{f}(\xi) d\xi = f_1(x). \quad (45)$$

Since $|e^{2\pi i(x,\xi)} \sum_{s \in \mathbb{Z}^d} \theta(M^{*-j} \xi + s) e^{2\pi i(M^j x, s)}| \leq C_1$, where C_1 depends only on φ , using (44) for f_2 and (45), we have

$$\begin{aligned} \left| f(x) - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf(M^{-j} \cdot)(-k) \varphi_{jk}(x) \right| &= \left| f_2(x) - m^{-j/2} \sum_{k \in \mathbb{Z}^d} Lf_2(M^{-j} \cdot)(-k) \varphi_{jk}(x) \right| \leq \\ \int_{|M^{*-j} \xi| \geq \delta} \left(1 + C_1 \sum_{[\beta] \leq N} |\alpha_\beta (2\pi i M^{*-j} \xi)^\beta| \right) |\widehat{f}(\xi)| d\xi &\leq C \|M^{*-j}\|^N \int_{|M^{*-j} \xi| \geq \delta} |\xi|^N |\widehat{f}(\xi)| d\xi, \end{aligned}$$

which yields the first inequality in (43). For the second inequality it remains to apply Lemma 15 with $\gamma = N$, $q = 1$. \diamond

Corollary 17 *If, under assumptions (a) – (c) of Theorem 16, a function f is such that its Fourier transform is supported in $\{\xi : |M^{*-j}\xi| < \delta\}$, where δ is from Definition 5, then*

$$f(x) = Lf(M^{-j}\cdot)(-k)\varphi_{jk}(x) \quad \text{for all } x \in \mathbb{R}^d. \quad (46)$$

*If, moreover, $\widehat{\varphi}$ is supported in $[-1/2, 1/2]^d$ and $\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(\xi)} = 1$ for all $\xi \in [-1/2, 1/2]^d$, then (46) holds for every f whose Fourier transform is supported in $M^{*j}[-1/2, 1/2]^d$.*

Proof. The first statement is a trivial consequence of Theorem 16. Analysing the proof of this theorem, it is easy to see that the second statement is also true. \diamond

Corollary 17 is an analog of the classical sampling theorem.

Now consider the following differential equation

$$Lf := \sum_{[\beta] \leq N} a_\beta D^\beta f = g,$$

where $a_0 \neq 0$ and g is a function band-limited to $[-1/2, 1/2]^d$. Let $\widetilde{\varphi}$ is a distribution associated with L . Assume that $\widetilde{\varphi}$ does not vanish on $[-1/2, 1/2]^d$ and define φ by

$$\widehat{\varphi}(\xi) = \begin{cases} \frac{1}{\widetilde{\varphi}(x)}, & \text{if } \xi \in [-1/2, 1/2]^d, \\ 0, & \text{if } \xi \notin [-1/2, 1/2]^d. \end{cases}$$

By Corollary 17, we have

$$f(x) = \sum_{k \in \mathbb{Z}^d} g(k)\varphi(x+k).$$

For example, if $d = 1$, $Lf = f - f''$, then $\varphi(x) = \int_{-\pi}^{\pi} \frac{\cos tx}{1+t^2} dt$. The latter function can be expressed via four special functions: hyperbolic sine, hyperbolic cosine, integral sine and integral cosine, see Wolfram Mathematica.

5 Falsified sampling expansions

If exact sampled values of a signal f are known, then sampling expansions are very useful for applications. Theorems 13, 14, and 16 provide error estimates for this case. Now we discuss what happens if exact sampled values $f(-M^{-j}k)$ are replaced by average values. We assume that at each point $M^{-j}k$ one knows the following average value of f

$$\frac{1}{V_{h(u)}} \int_{B_{h(u)}} f(M^{-j}k + M^{-j}t) dt = \frac{m^j}{V_{h(u)}} \int_{M^{-j}B_{h(u)}} f(M^{-j}k + t) dt =: \text{Av}_{h(u)}(f, M^{-j}k), \quad (47)$$

where $h(u)$ is a positive function defined on $(0, \infty)$ and u is a random value with probability density w ; V_h is the volume of the ball B_h . Set

$$E(f, M^{-j}k) = E(f, M^{-j}k, h, w) = \int_0^\infty du w(u) \text{Av}_{h(u)}(f, M^{-j}k).$$

Note that if $h(u) \equiv h > 0$, then $E(f, M^{-j}k) = \text{Av}_h(f, M^{-j}k)$.

We are interested in error analysis for the *falsified sampling expansions*

$$m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk}.$$

First we prove the following lemma.

Lemma 18 *Let $N \in \mathbb{N}$, let a function f be continuously differentiable up to order N , and let A be a real $d \times d$ matrix. Then for all $t, x \in \mathbb{R}^d$*

$$\sum_{[\beta] < N+1} \frac{D^\beta f(Ax)}{\beta!} (At)^\beta = \sum_{[\beta] < N+1} \frac{D^\beta f(A \cdot)(x)}{\beta!} t^\beta.$$

Proof. First we introduce some additional notations. Let $r \in \mathbb{Z}_+$, $O_r = \{\beta \in \mathbb{Z}_+^d : [\beta] = r\}$. Assume that the set O_r is ordered by lexicographic order. Namely, $(\beta_1, \dots, \beta_d)$ is less than $(\alpha_1, \dots, \alpha_d)$ in lexicographic order if $\beta_j = \alpha_j$ for $j = 1, \dots, i-1$ and $\beta_i < \alpha_i$ for some i . Let $S(A, r)$ be a $(\#O_r) \times (\#O_r)$ matrix which is uniquely determined by

$$\frac{(At)^\alpha}{\alpha!} = \sum_{\beta \in O_r} [S(A, r)]_{\alpha, \beta} \frac{t^\beta}{\beta!}, \quad (48)$$

where $\alpha \in O_r$, $t \in \mathbb{R}^d$. It can be verified that

$$\alpha! [S(A, r)]_{\alpha, \beta} = \beta! [S(A^*, r)]_{\beta, \alpha}. \quad (49)$$

The above notation and the latter fact is borrowed from [11].

Fix $\beta \in \mathbb{Z}_+^d$. Let \mathcal{E} be the set of ordered samples with replacement of size $[\beta]$ from the set $\{e_1, \dots, e_d\}$, where e_k is the k -th ort in \mathbb{R}^d . An element $e \in \mathcal{E}$ is a set $\{e_{i_1}, \dots, e_{i_{[\beta]}}\}$, where $i_l \in \{1, \dots, d\}$, $l = 1, \dots, [\beta]$, $\#\mathcal{E} = d^{[\beta]}$. For $e \in \mathcal{E}$ denote by $(e)_l := e_{i_l}$, $l = 1, \dots, [\beta]$. Let T be a function defined on \mathcal{E} by $T(e) := \sum_{i=1}^{[\beta]} (e)_i$. Note that $T(e) \in \mathbb{Z}_+^d$ and $[T(e)] = [\beta]$. Denote by b an element of \mathcal{E} so that $T(b) = \beta$. Such b is unique up to a permutation. Using the higher chain rule, we have

$$D^\beta f(A \cdot)(x) = \frac{\partial^{[\beta]} f(A \cdot)(x)}{\partial x^\beta} = \sum_{e \in \mathcal{E}} \frac{\partial^{[\beta]} f(y)}{\partial y^{T(e)}} \Big|_{y=Ax} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}}, \quad x \in \mathbb{R}^d,$$

where $\prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}}$ does not depend on x and $D^\beta f(A \cdot)(x)$ does not depend on the choice of b . For different elements $e, h \in \mathcal{E}$, we may have $T(e) = T(h)$. Thus, we can group terms in the sum with equal values of $T(\cdot)$. Namely,

$$D^\beta f(A \cdot)(x) = \sum_{\alpha \in \mathbb{Z}_+^d, [\alpha] = [\beta]} \frac{\partial^{[\beta]} f(y)}{\partial y^\alpha} \Big|_{y=Ax} \sum_{e \in \mathcal{E}, T(e) = \alpha} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}}. \quad (50)$$

If $f(x) = e^{2\pi i(t, x)}$, $t \in \mathbb{R}^d$, then $D^\beta f(A \cdot)(x) = D^\beta e^{2\pi i(A^* t, x)} = (A^* t)^\beta e^{2\pi i(t, Ax)}$. On the other hand, by (50),

$$D^\beta f(A \cdot)(x) = e^{2\pi i(t, Ax)} \sum_{\alpha \in \mathbb{Z}_+^d, [\alpha] = [\beta]} t^\alpha \sum_{e \in \mathcal{E}, T(e) = \alpha} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}}.$$

Thus, due to (48) with the matrix A replaced by A^* , we obtain

$$\sum_{e \in \mathcal{E}, T(e) = \alpha} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}} = [S(A^*, r)]_{\beta, \alpha} \frac{\beta!}{\alpha!}. \quad (51)$$

Now let f be an arbitrary function continuously differentiable up to order N , $0 \leq r \leq N$, $r \in \mathbb{Z}_+$. It follows from (49) and (51) that

$$\begin{aligned} \sum_{\alpha \in O_r} \frac{D^\alpha f(Ax)}{\alpha!} (At)^\alpha &= \sum_{\alpha \in O_r} D^\alpha f(Ax) \sum_{\beta \in O_r} [S(A, r)]_{\alpha, \beta} \frac{t^\beta}{\beta!} = \sum_{\beta \in O_r} t^\beta \sum_{\alpha \in O_r} D^\alpha f(Ax) \frac{[S(A^*, r)]_{\beta, \alpha}}{\alpha!} = \\ &= \sum_{\beta \in O_r} \frac{t^\beta}{\beta!} \sum_{\alpha \in O_r} D^\alpha f(Ax) \sum_{e \in \mathcal{E}, T(e)=\alpha} \prod_{i=1}^{[\beta]} \frac{\partial (Ax)^{(e)_i}}{\partial x^{(b)_i}} = \sum_{\beta \in O_r} \frac{t^\beta}{\beta!} D^\beta f(A \cdot)(x). \end{aligned}$$

It remains to sum the latter expression over r from 0 to N . \diamond

Let

$$Lf(M^{-j} \cdot)(k) = \sum_{[\beta] < N+1} a_\beta D^\beta f(M^{-j} \cdot)(k), \quad a_\beta = \int_0^\infty du \frac{w(u)}{\beta! V_{h(u)}} \int_{B_{h(u)}} t^\beta dt. \quad (52)$$

By Lemma 18, we have

$$\begin{aligned} Lf(M^{-j} \cdot)(k) &= \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{B_{h(u)}} \sum_{[\beta] < N+1} \frac{D^\beta f(M^{-j} \cdot)(k)}{\beta!} t^\beta dt = \\ &= \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{B_{h(u)}} \sum_{[\beta] < N+1} \frac{D^\beta f(M^{-j} k)}{\beta!} (M^{-j} t)^\beta dt = \\ &= m^j \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{M^{-j} B_{h(u)}} \sum_{[\beta] < N+1} \frac{D^\beta f(M^{-j} k)}{\beta!} t^\beta dt, \end{aligned} \quad (53)$$

and, due to the Tailor formula,

$$m^j \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{M^{-j} B_{h(u)}} f(M^{-j} k + t) dt \approx Lf(M^{-j} \cdot)(k).$$

To investigate the convergence and approximation order of falsified sampling expansions we can use Theorems 13, 14, 16, and estimate the sum $m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk}$, where

$$\varepsilon_j(k) := \int_0^\infty w(u) \text{Av}_{h(u)}(f, M^{-j} k) du - Lf(M^{-j} \cdot)(k). \quad (54)$$

Proposition 19 *Let $d < p \leq \infty$, $\varphi \in \mathcal{L}_p$ or $\varphi \in \mathcal{B}$, $p \neq \infty$. Suppose $N \in \mathbb{N}$, $f \in W_p^{N+1}$, the operator L is defined by (52), $\varepsilon_j(k)$ is defined by (54), w and h are as in (47). Then*

$$\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq C \int_0^\infty w(u) (1 + h^{N+1}(u)) du \|f\|_{W_p^{N+1}} \vartheta^{-j(N+1)} \quad (55)$$

for every positive number ϑ which is smaller in module than any eigenvalue of M and some C which does not depend on f , h , w , and j .

Proof. Let us fix $j \in \mathbb{N}$, $\varepsilon_j = \{\varepsilon_j(k)\}_{k \in \mathbb{Z}^d}$. If $\varphi \in \mathcal{L}_p$, then, due to Proposition 1,

$$\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p = m^{-j/p} \left\| \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{0k} \right\|_p \leq m^{-j/p} \|\varphi\|_{\mathcal{L}_p} \|\varepsilon_j\|_{\ell_p}. \quad (56)$$

If $\varphi \in \mathcal{B}$ and $p \neq \infty$, then, due to Proposition 9,

$$\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_p = m^{-j/p} \left\| \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{0k} \right\|_p \leq m^{-j/p} C_{\varphi, q} \|\varepsilon_j\|_{\ell_p}. \quad (57)$$

By the Taylor formula with integral remainder, we have

$$f(M^{-j}k + t) = \sum_{[\beta] < N+1} \frac{D^\beta f(M^{-j}k)}{\beta!} t^\beta + \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} t^\beta \int_0^1 (1-\tau)^N D^\beta f(M^{-j}k + t\tau) d\tau.$$

It follows from (53) that

$$\varepsilon_j(k) = m^j \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{M^{-j}B_{h(u)}} dt \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} t^\beta \int_0^1 (1-\tau)^N D^\beta f(M^{-j}k + t\tau) d\tau.$$

Hence, taking into account that $|t^\beta| \leq |t|^{[\beta]}$, we get

$$\begin{aligned} |\varepsilon_j(k)| &\leq \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} m^j \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{M^{-j}B_{h(u)}} dt |t|^{[\beta]} \int_0^1 |D^\beta f(M^{-j}k + t\tau)| d\tau \leq \\ &\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} m^j \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_0^1 d\tau \int_{\tau M^{-j}B_{h(u)}} \frac{|t|^{[\beta]}}{\tau^{[\beta]}} |D^\beta f(M^{-j}k + t)| \frac{dt}{\tau^d} = \\ &\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_0^1 d\tau \int_{\tau B_{h(u)}} |M^{-j}t|^{[\beta]} |D^\beta f(M^{-j}k + M^{-j}t)| \frac{dt}{\tau^{[\beta]+d}}. \end{aligned}$$

The latter integration is taken over the set $\{\tau \in [0, 1], t \in \tau B_{h(u)}\} = \{\tau \in [0, 1], |t| \leq \tau h(u)\}$, or equivalently $\{|t| \in [0, h(u)], \frac{|t|}{h(u)} \leq \tau \leq 1\}$. Changing the order of integration, we obtain

$$\begin{aligned} |\varepsilon_j(k)| &\leq \sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{N+1}{\beta!} \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{B_{h(u)}} |M^{-j}t|^{[\beta]} |D^\beta f(M^{-j}k + M^{-j}t)| dt \int_{\frac{|t|}{h(u)}}^1 \frac{d\tau}{\tau^{N+d+1}} \leq \\ &\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{(N+1)}{(N+d)\beta!} \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{B_{h(u)}} |M^{-j}t|^{N+1} |D^\beta f(M^{-j}k + M^{-j}t)| \left(\frac{h(u)}{|t|}\right)^{N+d} dt \leq \\ &\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{\|M^{-j}\|^{N+1}}{\beta!} \int_0^\infty du \frac{w(u)h^{N+d}(u)}{V_{h(u)}} \int_{B_{h(u)}} \frac{|D^\beta f(M^{-j}k + M^{-j}t)|}{|t|^{d-1}} dt = \\ &\sum_{\beta \in \mathbb{Z}_+^d, [\beta] = N+1} \frac{\|M^{-j}\|^{N+1} m^j}{\beta!} \int_0^\infty du \frac{w(u)h^N(u)}{V_1} \int_{M^{-j}B_{h(u)}} \frac{|D^\beta f(M^{-j}k + t)|}{|M^j t|^{d-1}} dt. \end{aligned}$$

If $p = \infty$, then

$$\int_{M^{-j}B_{h(u)}} \frac{|D^\beta f(M^{-j}k+t)|}{|M^j t|^{d-1}} dt \leq \int_{M^{-j}B_{h(u)}} \frac{dt}{|M^j t|^{d-1}} \|f\|_{W_\infty^{N+1}}.$$

If $p \neq \infty$, then using Hölder's inequality, we have

$$\int_{M^{-j}B_{h(u)}} \frac{|D^\beta f(M^{-j}k+t)|}{|M^j t|^{d-1}} dt \leq \left(\int_{M^{-j}B_{h(u)}} |D^\beta f(M^{-j}k+t)|^p dt \right)^{1/p} \left(\int_{M^{-j}B_{h(u)}} \frac{dt}{|M^j t|^{q(d-1)}} \right)^{1/q},$$

where $q = \frac{p}{p-1}$. Since

$$\int_{M^{-j}B_{h(u)}} \frac{dt}{|M^j t|^{q(d-1)}} = m^{-j} \int_{B_{h(u)}} \frac{dt}{|t|^{q(d-1)}} = m^{-j} h^{-q(d-1)+d}(u) \int_{B_1} \frac{dt}{|t|^{q(d-1)}}$$

and $q(d-1) < d$, the latter integral is finite. Summarizing the above estimates, we obtain

$$|\varepsilon_j(k)|^p \leq C_1 m^j \|M^{-j}\|^{p(N+1)} \int_0^\infty du w(u) h^{p(N-d+1+d/q)}(u) \int_{M^{-j}k+M^{-j}B_{h(u)}} \sum_{\beta \in \mathbb{Z}_+^d, |\beta|=N+1} |D^\beta f(t)|^p dt$$

and

$$\sum_{k \in \mathbb{Z}^d} |\varepsilon_j(k)|^p \leq C_2 m^j \|M^{-j}\|^{p(N+1)} \int_0^\infty du w(u) h^{p(N-d+1+d/q)}(u) (1 + h^d(u)) \|f\|_{W_p^{N+1}}^p,$$

where C_2 does not depend on f , h , and j . It follows that

$$\sum_{k \in \mathbb{Z}^d} |\varepsilon_j(k)|^p \leq C_2 m^j \|M^{-j}\|^{p(N+1)} \|f\|_{W_p^{N+1}}^p \int_0^\infty w(u) (h^{p(N-d+1+d/q)}(u) + h^{p(N+1)}(u)) du.$$

Combining this with (56), (57) and (6), we get (55). \diamond

Proposition 20 *Let $d < p' < \infty$, $\varphi \in \mathcal{B}$. Suppose $N \in \mathbb{N}$, $f \in W_{p'}^{N+1}$, operator L is defined by (52), $\varepsilon_j(k)$ is defined by (54), w and h are as in (47). Then*

$$\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_\infty \leq C \int_0^\infty w(u) (1 + h^{N+1}(u)) du \|f\|_{W_{p'}^{N+1}} \vartheta^{-j(N+1)} \quad (58)$$

for every positive number ϑ which is smaller in module than any eigenvalue of M and some C which does not depend on f , h , w and j .

Proof. Let us fix $j \in \mathbb{N}$, $\varepsilon_j = \{\varepsilon_j(k)\}_{k \in \mathbb{Z}^d}$. Because of (22), it is easy to see that

$$\sum_{k \in \mathbb{Z}^d} |\varphi(x+k)|^q < C'_{\varphi,q}$$

for every $q > 1$ and every $x \in \mathbb{R}^d$. It follows that

$$\left\| m^{-j/2} \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi_{jk} \right\|_\infty = \sup_{x \in \mathbb{R}^d} \left| \sum_{k \in \mathbb{Z}^d} \varepsilon_j(-k) \varphi(M^j x + k) \right| \leq (C'_{\varphi,q'})^{1/q'} \|\varepsilon_j\|_{\ell_{p'}},$$

where $1/q' + 1/p' = 1$. To get (58) it remains to repeat all arguments of the proof of Theorem 19 after relation (57), replacing p and q by p' and q' respectively. \diamond

Remark 21 If M is an isotropic matrix for which λ is an eigenvalue, then in the proof of Propositions 19, 20 we can use inequality (8) instead of (6). Hence ϑ in (55) and (58) can be replaced by λ .

Using the above results we can state the convergence and approximation order of falsified sampling expansions.

Theorem 22 Let $d < p \leq \infty$, $p \geq 2$, $1/p + 1/q = 1$, $N \in \mathbb{Z}_+$, M be an isotropic matrix dilation and λ be its eigenvalue. Suppose φ and n are as of Theorem 13, where $\tilde{\varphi}$ is the distribution associated with the differential operator L given by (52), w and h are as in (47) and

$$\int_0^\infty w(u)(1 + h^{N+1}(u)) du < \infty;$$

$f \in L_p$, $\hat{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-1-\frac{d}{q}-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$. Then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} C|\lambda|^{-j(N+1)} & \text{if } n > N+1 \\ C|\lambda|^{-jn} & \text{if } n \leq N+1 \end{cases}, \quad (59)$$

where C does not depend on j .

Proof. The proof follows from Theorem 13, item (B), Proposition 19, and Remark 21. \diamond

Theorem 23 Let $d < p \leq \infty$, $p \geq 2$, $1/p + 1/q = 1$, $N \in \mathbb{Z}_+$. Suppose φ is as in item (b) of Theorem 13 or $\varphi \in \mathcal{B}$, $p \neq \infty$; $\tilde{\varphi}$ is the distribution associated with the differential operator L given by (52), $\tilde{\varphi}$ and φ are strictly compatible; w and h are as in Theorem 22; $f \in L_p$, $\hat{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-1-\frac{d}{q}-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$. Then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq C\vartheta^{-j(N+1)} \quad (60)$$

for every positive number ϑ which is smaller in module than any eigenvalue of M and some C which does not depend on j .

Proof. The proof follows from Theorem 13, item (C), Theorem 14, and Proposition 19. \diamond

Remark 24 If the assumption on f in Theorems 22 and 23 is replaced by $f \in W_p^{N+1}$, $\hat{f} \in L_q$, $\hat{f}(\xi) = O(|\xi|^{-N-1-\frac{d}{q}})$ as $|\xi| \rightarrow \infty$, then Theorem 23 remains to be true, and inequality (59) in Theorem 22 must be replaced by

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} C|\lambda|^{-j(N+1)} & \text{if } n > N+1 \\ C(j+1)^{\frac{1}{q}}|\lambda|^{-j(N+1)} & \text{if } n = N+1 \\ C|\lambda|^{-jn} & \text{if } n \leq N+1 \end{cases}.$$

Theorem 25 Let $d < p' < \infty$, $1/p' + 1/q' = 1$, $N \in \mathbb{Z}_+$. Suppose $\varphi \in \mathcal{B}$, $\tilde{\varphi}$ is the distribution associated with the differential operator L given by (52), $\tilde{\varphi}$ and φ are strictly compatible; w and h are as in Theorem 22; $f \in L_{p'}$, $\hat{f} \in L_{q'}$, $\hat{f}(\xi) = O(|\xi|^{-N-1-\frac{d}{q'}-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$. Then

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_\infty \leq C\vartheta^{-j(N+1)} \quad (61)$$

for every positive number ϑ which is smaller in module than any eigenvalue of M and some C which does not depend on j .

Proof. The proof follows from Theorem 16 and Proposition 20. \diamond

Remark 26 If the assumption on f in Theorem 25 is replaced by $f \in W_{p'}^{N+1}$, $\widehat{f} \in L_1$, $\widehat{f}(\xi) = O(|\xi|^{-N-d-\varepsilon})$ as $|\xi| \rightarrow \infty$, $\varepsilon > 0$, then Theorem 23 remains to be true with

$$\left\| f - m^{-j/2} \sum_{k \in \mathbb{Z}^d} E(f, M^{-j}k) \varphi_{jk} \right\|_{\infty} \leq C \vartheta^{-j(N+\min(1,\varepsilon))}$$

instead of 61.

Observing the proof of Proposition 19, one can see that in the one-dimensional case an analog of (55) holds true for a wider class of functions f and any $p \geq 1$. Indeed, in this case we have

$$\begin{aligned} |\varepsilon_j(k)| &\leq \frac{1}{(N+1)!} \int_0^\infty du \frac{w(u)}{V_{h(u)}} \int_{B_{h(u)}} |M^{-j}t|^{N+1} |f^{(N+1)}(M^{-j}k + M^{-j}t)| \left(\frac{h(u)}{|t|} \right)^{N+1} dt \leq \\ &\frac{M^{-j(N+1)}}{(N+1)!} \int_0^\infty du \frac{w(u)h^{N+1}(u)}{V_{h(u)}} \int_{B_{h(u)}} |f^{(N+1)}(M^{-j}k + M^{-j}t)| dt = \\ &\frac{M^{-jN}}{(N+1)!} \int_0^\infty du \frac{w(u)h^{N+1}(u)}{V_{h(u)}} \int_{M^{-j}B_{h(u)} + M^{-j}k} |f^{(N+1)}(t)| dt. \end{aligned}$$

It follows that

$$\left\| m^{-j/2} \sum_{k \in \mathbb{Z}} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq m^{-j/p} \|\varphi\|_p \sum_{k \in \mathbb{Z}} |\varepsilon_j(k)| \leq C \int_0^\infty w(u)h^N(u) du \|f\|_{W_1^{N+1}} M^{-j(N+\frac{1}{p})}.$$

This yields the following statements.

Proposition 27 Let $d = 1$, $p \geq 1$, $N \in \mathbb{N}$, $\varphi \in L_p$. Suppose $f \in W_1^{N+1}$, $\varepsilon_j(k)$ is defined by (54), w and h are as in (47) and

$$\int_0^\infty w(u)(1+h^N(u)) du < \infty.$$

Then

$$\left\| M^{-j/2} \sum_{k \in \mathbb{Z}} \varepsilon_j(-k) \varphi_{jk} \right\|_p \leq C \|f\|_{W_1^{N+1}} M^{-j(N+\frac{1}{p})}, \quad (62)$$

where C does not depend on f and j .

Theorem 28 Let $d = 1$, $2 \leq p \leq \infty$, $N \in \mathbb{Z}_+$. Suppose φ and n are as in of Theorem 13, where $\tilde{\varphi}$ is the distribution associated with the differential operator L given by (52), or $\varphi \in \mathcal{B}$; w and h are as in Proposition 27; $f \in W_1^{N+1}$, $f^{(N+1)} \in \text{Lip}_{L_1} \varepsilon$, $\varepsilon > 0$. Then

$$\left\| f - M^{-j/2} \sum_{k \in \mathbb{Z}} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq \begin{cases} CM^{-j(N+\frac{1}{p})} & \text{if } n > N + \frac{1}{p}, \\ CM^{-jn} & \text{if } n \leq N + \frac{1}{p}, \end{cases} \quad (63)$$

where C does not depend on j .

Theorem 29 Let $d = 1, 2 \leq p \leq \infty, N \in \mathbb{Z}_+$. Suppose φ is as in item (b) of Theorem 13 or $\varphi \in \mathcal{B}$; $\tilde{\varphi}$ is the distribution associated with the differential operator L given by (52), $\tilde{\varphi}$ and φ are strictly compatible; w and h are as in Proposition 27; $f \in W_1^{N+1}, f^{(N+1)} \in \text{Lip}_{L_1} \varepsilon, \varepsilon > 0$. Then

$$\left\| f - M^{-j/2} \sum_{k \in \mathbb{Z}} E(f, M^{-j}k) \varphi_{jk} \right\|_p \leq CM^{-j(N+\frac{1}{p})} \quad (64)$$

for some C which does not depend on j .

6 Examples

In this section some examples will be given to illustrate the above results.

I. First we discuss construction of band-limited functions φ . For every differential operator L one can easily construct φ supported on a small neighborhood of zero and such that on the same neighborhood $\hat{\varphi}\tilde{\varphi} = 1$, where $\tilde{\varphi}$ is a distribution associated with L . The function φ is in \mathcal{B} , $\tilde{\varphi}$ and φ are strictly compatible, so, due to Theorems 14 and 23, the approximation order of the corresponding differential and falsified expansions with arbitrary matrix dilation depends only on how smooth is a function f . In particular, in the case $Lf = f$, the function $\varphi(x) = \prod_{k=1}^d \frac{\sin \pi x_k}{\pi x_k}$ can be taken as φ . Note that this function φ is appropriate not only for the expansions with exact sampled values, but also for falsified expansions. Indeed, if $N = 2$, then $Lf = f + \sum_{[\beta]=1} a_\beta D^\beta f$, where $a_\beta = 0$ by (52). Thus φ satisfies all conditions of Theorem 23 with $N = 1$, and, according to (60), the approximation order of the corresponding falsified sampling expansions is 2 for smooth enough functions f .

Hence, theoretically, we have a simple and very good solution to our problem. However such expansions are not good from the computational point of view because in the case $Lf \neq f$ we will not be able to derive explicit formulas which is needed for implementations. \diamond

II. Let

$$\hat{\varphi}(\xi) = \prod_{k=1}^d \left(\frac{\sin \pi \xi_k}{\pi \xi_k} \right)^2.$$

Since $\varphi(x) = \prod_{k=1}^d (1 - |x_k|) \chi_{[-1,1]^d}(x)$, φ is compactly supported and in \mathcal{L}_p . Also, the function $\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q$ is bounded, $\hat{\varphi}$ is continuously differentiable up to any order, the function $\sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \hat{\varphi}(\xi + l)|$ is bounded near the origin for $[\beta] = 2$. Also, the Strang-Fix condition of order 2 holds for φ . The values of $\hat{\varphi}$ and its derivatives at the origin are

$$\hat{\varphi}(\mathbf{0}) = 1, \quad D^\beta \hat{\varphi}(\mathbf{0}) = 0, \quad [\beta] = 1.$$

So, if $\tilde{\varphi} = \delta$, then all assumptions of Theorem 13 are satisfied. The corresponding sampling expansion of a signal f interpolates f at the points $M^{-j}k, k \in \mathbb{Z}^d$, the approximation order depends on how smooth is f , but, according to (41), it cannot be better than 2. Again $\tilde{\varphi}$ is associated with the differential operator $Lf = f + \sum_{[\beta]=1} a_\beta D^\beta f$, where $a_\beta = 0$. Hence the functions $\varphi, \tilde{\varphi}$ satisfy all conditions of Theorem 22 with $n = 2, N = 1$, and, according to (59), the approximation order of the corresponding falsified sampling expansions is 2 for smooth enough functions f . \diamond

III. Let $d = 2$,

$$\hat{\varphi}(\xi_1, \xi_2) = \frac{1}{(\pi^2 \xi_1 \xi_2)^3} (\sin^3 \pi \xi_1 \sin^3 \pi \xi_2 + b_1 \sin^3 \pi \xi_1 \sin^4 \pi \xi_2 + b_2 \sin^4 \pi \xi_1 \sin^3 \pi \xi_2).$$

Again φ is compactly supported and belongs to \mathcal{L}_p . Also, the function $\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + k)|^q$ is bounded, $\hat{\varphi}$ is continuously differentiable up to any order. Since the trigonometric polynomial in the numerator

of $\widehat{\varphi}$ is bounded, the function $\sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \widehat{\varphi}(\xi + l)|$ is bounded near the origin for $[\beta] = 3$. Also, the Strang-Fix condition of order 3 holds for φ . The values of $\widehat{\varphi}$ and its derivatives at the origin are

$$\widehat{\varphi}(0, 0) = 1, \quad D^{(1,0)} \widehat{\varphi}(0, 0) = D^{(0,1)} \widehat{\varphi}(0, 0) = 0,$$

$$D^{(2,0)} \widehat{\varphi}(0, 0) = \pi^2(2b_1 - 1), \quad D^{(0,2)} \widehat{\varphi}(0, 0) = \pi^2(2b_2 - 1), \quad D^{(1,1)} \widehat{\varphi}(0, 0) = 0.$$

Now, we choose an appropriate differential operator L in the form $Lf = f + a_{(2,0)} D^{(2,0)} f + a_{(0,2)} D^{(0,2)} f$, or equivalently, the associated distribution $\widetilde{\varphi} = \delta + \overline{a_{(2,0)}} D^{(2,0)} \delta + \overline{a_{(0,2)}} D^{(0,2)} \delta$. Since $\widehat{\widetilde{\varphi}}(\xi) = 1 - 4\pi^2 \overline{a_{(2,0)}} \xi_1 - 4\pi^2 \overline{a_{(0,2)}} \xi_2$, we have

$$\widehat{\widetilde{\varphi}}(0, 0) = 1, \quad D^{(2,0)} \widehat{\widetilde{\varphi}}(0, 0) = -4\pi^2 \overline{a_{(2,0)}}, \quad D^{(0,2)} \widehat{\widetilde{\varphi}}(0, 0) = -4\pi^2 \overline{a_{(0,2)}},$$

$$D^{(1,0)} \widehat{\widetilde{\varphi}}(0, 0) = D^{(0,1)} \widehat{\widetilde{\varphi}}(0, 0) = D^{(1,1)} \widehat{\widetilde{\varphi}}(0, 0) = 0.$$

To satisfy condition $D^\beta (1 - \widehat{\widetilde{\varphi}})(0, 0) = 0$ for $[\beta] < 3$, we have to provide

$$b_1 = \frac{1}{2}(1 - 4\overline{a_{(2,0)}}), \quad b_2 = \frac{1}{2}(1 - 4\overline{a_{(0,2)}}).$$

Finally, all conditions of Theorem 13 are satisfied. The approximation order depends on how smooth is f , but, according to (41), it cannot be better than $n = 3$.

Now we show that the coefficients b_1, b_2 can be chosen such that all conditions of Theorem 22 are satisfied with $n = 3, N = 2$. In this case the differential operator L is given by (52). The coefficients $a_\beta, [\beta] < 3$, are as follows

$$a_{0,0} = 1, \quad a_{1,0} = a_{0,1} = a_{1,1} = 0, \quad a_{2,0} = a_{0,2} = \frac{1}{8} \int_0^\infty h(x)^2 \omega(x) dx.$$

Thus, we set $b_1 = b_2 = \frac{1}{2}(1 - 4a_{(2,0)})$. According to (59), the approximation order of the corresponding falsified sampling expansions is 3 for smooth enough functions f . \diamond

IV. Let $d = 1$,

$$\widehat{\varphi}(\xi) = \frac{\sin^4 \pi \xi + b_1 \sin^5 \pi \xi + b_2 \sin^6 \pi \xi + b_3 \sin^7 \pi \xi}{(\pi \xi)^4}.$$

Since φ is bounded and compactly supported, it is in \mathcal{L}_p , and $\widehat{\varphi}$ is continuously differentiable up to any order. Also, $\sum_{k \in \mathbb{Z}^d} |\widehat{\varphi}(\xi + k)|^q$ is bounded. Since the trigonometric polynomial in the numerator of $\widehat{\varphi}$ is bounded, the function $\sum_{l \in \mathbb{Z}^d, l \neq 0} |D^\beta \widehat{\varphi}(\xi + l)|$ is bounded near the origin for $\beta = 4$. Also the Strang-Fix condition of order 4 holds for φ . The values of $\widehat{\varphi}$ and its derivatives at the origin are

$$\widehat{\varphi}(0) = 1, \quad \widehat{\varphi}'(0) = b_1 \pi, \quad \widehat{\varphi}''(0) = \frac{2}{3} \pi^2 (3b_2 - 2), \quad \widehat{\varphi}'''(0) = \pi^3 (6b_3 - 5b_1).$$

Now, we choose the appropriate differential operator L in the form $Lf = f + a_1 f' + a_2 f'' + a_3 f'''$, or equivalently, the associated distribution $\widetilde{\varphi} = \delta - \overline{a_1} \delta' + \overline{a_2} \delta'' - \overline{a_3} \delta'''$. Since $\widehat{\widetilde{\varphi}}(\xi) = 1 - 2\pi i \overline{a_1} \xi - 4\pi^2 \overline{a_2} \xi^2 + 8\pi^3 i \overline{a_3} \xi^3$, we have

$$\widehat{\widetilde{\varphi}}(0) = 1, \quad \widehat{\widetilde{\varphi}}'(0) = -2\pi i \overline{a_1}, \quad \widehat{\widetilde{\varphi}}''(0) = -8\pi^2 \overline{a_2}, \quad \widehat{\widetilde{\varphi}}'''(0) = 48\pi^3 i \overline{a_3}.$$

To satisfy condition $D^\beta (1 - \widehat{\widetilde{\varphi}})(0) = 0$ for $\beta = 0, 1, 2, 3$, we have to provide

$$\begin{aligned} (1 - \widehat{\widetilde{\varphi}})'(0) &= \pi(b_1 - 2i\overline{a_1}) = 0, \\ (1 - \widehat{\widetilde{\varphi}})''(0) &= \frac{2}{3} \pi^2 (-2 + 3b_2 - 12\overline{a_2} - 6i\overline{a_1} b_1) = 0, \\ (1 - \widehat{\widetilde{\varphi}})'''(0) &= -\pi^3 (5b_1 - 6b_3 + 4i(3b_2 - 2)\overline{a_1} + 24b_1 \overline{a_2} - 48i\overline{a_3}) = 0. \end{aligned} \tag{65}$$

Thus, the coefficients of the function $\widehat{\varphi}$ can be easily found using the coefficients of the differential operator L . Finally, all conditions of Theorem 13 are satisfied. The approximation order depends on how smooth is f , but, according to (41) with $|\lambda| = |M|$, it cannot be better than $n = 4$.

Now we show that the coefficients b_1, b_2, b_3 can be chosen such that all conditions of Theorem 22 are satisfied with $n = 4, N = 3$. In this case the differential operator $Lf = a_0f + a_1f' + a_2f'' + a_3f'''$ is given by (52) and its coefficients are defined as

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{6} \int_0^\infty h(x)^2 \omega(x) dx, \quad a_3 = 0.$$

Using (65), we set $b_1 = 0, b_2 = \frac{2}{3} + 4a_2, b_3 = 0$. According to (59), the approximation order of the corresponding falsified sampling expansions is 4 for smooth enough functions f .

Note that all conditions of Theorem 28 are also satisfied, which provides approximation order for a wider class of functions f . Namely, according to (63), the approximation order is $3 + \frac{1}{p}$, whenever $f \in W_1^4, f^{(IV)} \in \text{Lip}_{L_1} \varepsilon, \varepsilon > 0$. \diamond

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